Long Run Output, Welfare and Fertility in a Neoclassical OLG Growth Model: The Effects of Intra-Generational Policies

Luciano Fanti[*] and Luca Gori[**]

Preliminary Draft
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Abstract

This paper examines the role played by intra-generational tax policies on both long-run economic growth and lifetime welfare in a simple OLG model of neoclassical growth with endogenous fertility behaviour. We show that the introduction of such policies - although only involving the income of the young-adult generation in a purely redistributionary way - have important effects on demographic and macroeconomic variables as well as on the lifetime welfare of the representative generation. In particular, provided moderate (low) values of the capital's weight in technology, a welfare-maximising wage subsidy (tax) rate is picked up. Further, if the size of the capital's weight in technology relative to those of both the rate of time preference and the parents' preference for children parameters is high, then the wage subsidy rate may also be used for controlling population growth, trading off between higher welfare and lower fertility. Finally, we prove that a benevolent government (which plays a Stackelberg game in the decentralised economy) is able to achieve the optimal population growth rate, i.e. it behaves as if it were the social planner but allowing individuals to freely choose their optimal values of both periods consumption and the number of children. These results may have applications to government policies for economic growth and welfare.

Keywords: Taxes; Subsidies; Fertility; Neoclassical Economic Growth; Welfare

JEL Classification: D91; H24; J13; O41

[*] Professor of Economics, Department of Economics, University of Pisa, Via Cosimo Ridolfi, 10, I-56124 Pisa (PI), Tuscany, Italy; e-mail address: lfanti@ec.unipi.it; tel.: +39 050 22 16 369; fax: +39 050 22 16 384.

[**] Corresponding author. Ph.D. Student in Economics, Department of Economics, University of Pisa, Via Cosimo Ridolfi, 10, I-56124 Pisa (PI), Tuscany, Italy; e-mail address: luca.gori@ec.unipi.it; tel.: +39 050 22 16 372; fax: +39 050 22 16 384.

1 Introduction

The purpose of this paper is to investigate the long-run consequences of intra-generational tax policies on both economic growth[1] and welfare in a basic OLG model of neoclassical growth with endogenous fertility behaviour. While the effects of inter-generational transfers and taxes on economic growth and consumption in OLG frameworks have been largely studied dating back to the seminal Diamond (1965), less attention has been paid to the effects of intra-generational policies on the young people in a context in which agents choose optimally both the consumption/saving path and the number of children. In general, it has been sometimes supposed in literature that policies such as the reduction of taxes on labour income or the introduction of labour subsidies may have positive effects on capital accumulation (see, for instance, Tullio (1987), Begg and Portes (1993), Drèze and Malinvaud (1994), and Daveri and Tabellini (2000)). More recently Petrucci and Phelps (2005) showed - in an OLG small open economy model of neoclassical growth with continuous time and endogenous labour-leisure choice - that labour subsidies are neutral for the macroeconomic equilibrium, in the short as well as in the long-run.

In this paper we show - in a Diamond OLG frame with Cobb-Douglas technology and preferences - that proportional-to-wage labour subsidies (taxes), although financed by (used for the financing of) lump sum taxes (subsides) on the income of the young-adult individuals at balanced budget, always enhances (deteriorates) capital accumulation and output. Further, provided a sufficiently high/low capital's share in income, a welfare-maximising labour subsidy/tax is picked up. It should also be emphasised the role played by these intra-generational tax policies on the long-run population growth: although agents' fertility behaviours are independent of the wage rate, the subsidy (tax) rate on labour income plays a direct role in reducing (augmenting) the agents' optimal number of children.

It is worth noticing that in the original Diamond's model with exogenous fertility, while it is known that inter-generational transfers from the old people to the younger generation (e.g., higher taxes on the income from capital and lower taxes on labour at balanced budget) may lead to faster capital accumulation, intra-generational policies - involving only the income of the young-adult people in a purely redistributionary manner - taxing away income and rebating it as a labour subsidy (and conversely), seems to be "neutral" as regards both long run economic growth and welfare. However, this paper proves, in presence of endogenous fertility, that intra-generational policies are very "effective".

As regards fertility choices, we prove that the relation among the subsidy (tax) rate and the optimal number of children is always negative (positive), and that, when the capital's weight in technology is relatively high, the wage subsidy rate may be used by the policy maker for controlling population growth, trading off between higher welfare and lower fertility.

Further, "optimality" analysis reveals that a benevolent government (which plays a Stackelberg game in the decentralised economy) is able to achieve the optimal population growth rate, that is it behaves as it were the social planner but allowing individuals to freely chose their optimal values of both consumption levels and the number of children. Although, for instance for ethical reasons, a government does not pursue the goal of optimal population growth rate, but it only aims to maximise the utility index resulting by the optimal individuals' choices rather than the optimal social behaviours, it obtains, surprisingly, the golden rule number of children by introducing appropriate intra-generational tax policies.

These findings, at our knowledge, have not been so far explored in literature and constitute the value added of the present paper.

[1] In this paper the term economic growth always refers to the level (rather than to the rate of growth) of the long run income, according to the terminology of the neoclassical growth theory (e.g. Solow (1956) and Mankiw et al. (1992)). In any case, needless to say, an increase in the long run output level, implies a transitional increase in the rate of growth as well.

The paper is organised as follows: in Section 2 we present the model and we investigate the effects of the tax policies on capital, output, fertility as well as welfare by also providing a normative analysis. Concluding comments are in Section 3.

2 The Model

We characterise a simple neoclassical OLG growth model[2] - as in Diamond (1965) - with endogenous fertility and intra-generational tax policies. The economy is closed to international trade and goods, capital and labour markets are competitive. There is no uncertainty. In the sequel, we will show that seemingly neutral government interventions which redistribute resources among the younger generation have effects on both demographic and macroeconomic variables as well as on the lifetime welfare of the representative generation. The model is outlined as follows.

2.1 Individuals. Agents have identical preferences and are assumed to belong to an overlapping generations structure with finite lifetimes. Life is separated among three periods: childhood, young adulthood and old-age. During childhood individuals do not make any decisions. Adult individuals belonging to generation, say, t have a homothetic and separable utility function defined over consumption when young ($c_t^{\,y}$) and old ($c_{t+1}^{\,o}$) and from having children (n_t),[3] as in Galor and Weil (1996).[4] We suppose that rearing children requires a time cost $q \in (0,1)$ per child. This hypothesis results in an endogenous supply of labour: there exists, in fact, a trade-off between working in the labour market and raising children. Only young-adult individuals (N_t) join the labour force devoting a fraction $h_t = 1 - qn_t$ of their time to work in the labour market with qn_t being the share of time spent to raising children. Each worker receives the market-clearing wage (w_t) plus (minus) a proportional-to-wage labour subsidy (tax) at the constant rate $\sigma > 0$ ($-1 < \sigma < 0$). In addition, 1) if $\sigma > 0$, lump-sum taxes ($\tau_t > 0$) are levied by the government and used to finance the wage benefit system at balanced budget (tax-cum-subsidy policy, T/S henceforth); and 2) if $-1 < \sigma < 0$, the labour tax is rebated in the economy and used for the financing of a lump-sum subsidy scheme ($\tau_t < 0$) on the income of the younger generation at balanced budget (subsidy-cum-tax policy, S/T henceforth). The total income of each young-adult individual is used to consume and to save. During old-age agents are retired and live on the proceeds of their savings (s_t) plus the accrued interest at the rate r_{t+1}. The representative individual born at time t is faced with the problem of maximising the following logarithmic felicity function:

$$\max_{\{c_t^{\,y}, c_{t+1}^{\,o}, n_t\}} U_t\left(c_t^{\,y}, c_{t+1}^{\,o}, n_t\right) = (1-\phi)\ln\left(c_t^{\,y}\right) + \phi \ln\left(c_{t+1}^{\,o}\right) + \rho \ln(n_t),$$

subject to

$$c_t^{\,y} + c_{t+1}^{\,o}/(1+r_{t+1}) = w_t(1+\sigma)(1-qn_t) - \tau_t,$$

where $\phi \in (0,1)$ captures the relative weight of both periods consumption (with $\phi/(1-\phi)$ being the rate of time preference), and $\rho > 0$ represents the parents' preference for children parameter. The

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[2] Two reference textbooks are Azariadis (1993) and De La Croix and Michel (2002).

[3] Note that the number of children is $n_t \geq 1$ with $n_t - 1$ being the population growth rate (for simplicity, the mortality rate has not been included in the analysis). Some authors, including Samuelson (1975), used $N_{t+1}/N_t = 1+n$ with n indicating the rate of population growth. Our approach is used in most papers with endogenous fertility (see, for instance, Abio (2003)).

[4] Since one scope of this paper is to isolate the relation among individuals' fertility behaviour and tax policies, as a first attempt we ignore both the trade-off between child quantity and quality, and the assumption that parents maximise utility of their offsprings, which has been employed to explain economic growth and stagnation by - among others - Becker et al. (1990) and Ehrlich and Lui (1991).

higher ϕ the more individuals smooth consumption over time, and the higher ρ the more parents are children-interested.
The first order conditions for an interior solution are:

$$c_{t+1}{}^{o} = \frac{\phi}{1-\phi}(1+r_{t+1})c_t{}^{y}, \tag{1}$$

$$n_t = \frac{\rho}{1-\phi}\cdot\frac{c_t^{\,y}}{qw_t(1+\sigma)}. \tag{2}$$

Eq. (1) equates the marginal utility of current and future consumption in terms of current consumption, whereas eq. (2) equates the marginal utility of having a child with the involved marginal costs in terms of forgone utility of consumption.
Thus, by using (1) and (2) together with the inter-temporal budget constraint, the optimal number of children and the savings path are given by:

$$n_t = \frac{\rho}{1+\rho}\cdot\frac{w_t(1+\sigma)-\tau_t}{qw_t(1+\sigma)}, \tag{3}$$

$$s_t = \frac{\phi}{1+\rho}\left[w_t(1+\sigma)-\tau_t\right]. \tag{4}$$

2.2 Firms. There are two factors of production: physical capital (K) and labour (L). The representative firm owns a Cobb-Douglas technology by which both inputs are transformed into consumption goods, that is $Y_t = AK_t{}^{\alpha}L_t{}^{1-\alpha}$,[5] where $A>0$ is a scale parameter and $\alpha\in(0,1)$ is the capital's weight in technology. It hires aggregate capital stock as well as labour ($L=hN$) - according to their marginal productivity - to maximise profits. Defining $k_t := K_t/N_t$ and $y_t := Y_t/N_t$ as capital and output per-capita respectively, the intensive form production function becomes:

$$y_t = Ak_t{}^{\alpha}h_t^{1-\alpha}. \tag{5}$$

Assuming that final output is traded at a unit price, profits maximisation leads to the following marginal conditions:[6]

$$r_t = \alpha A(k_t/h_t)^{\alpha-1}-1, \tag{6}$$

$$w_t = (1-\alpha)A(k_t/h_t)^{\alpha}. \tag{7}$$

2.3 Government. The government runs a balanced budget policy in each period. The proportional-to-wage labour subsidy (tax) - at the constant rate $\sigma>0$ ($-1<\sigma<0$) - is entirely financed by levying and adjusting over time (used for the financing of time varying) lump-sum taxes ($\tau_t>0$) (subsidies ($\tau_t<0$)) on the income of each young-adult individual such as to balance out the budget. Therefore, the time-t government constraint is simply:

$$\sigma w_t h_t = \tau_t. \tag{8}$$

Note one important feature of the present model: we have deliberately chosen taxation policies involving the income of the young-adult individuals in a purely redistributionary way only, that is income taxed away from the young turned back to the same individuals as a benefit for the hours of work (and conversely). This feature is important because in OLG models, as known dating back to

[5] Adding exogenous growth in labour productivity does not alter any of the substantive conclusions of the model and, hence, it is not included here.

[6] We have assumed that physical capital totally depreciates over time, i.e. $\delta=1$. This assumption is not unrealistic in the present context, because as noticed by De La Croix and Michel ((2002), p. 338) "even if one assumes a rather low annual depreciation rate of 5%, 79% of the stock of capital is depreciated after 30 years".

Bertola (1996) and Uhlig and Yanagawa (1996), taxes on the income from capital could lead to faster economic growth since all savings are performed by young agents. Then, in the present context, taxation policy does not cause any transfer from the old-age to the younger generation (as, instead, it would have been the case with capital income taxes); thus, the effects on both economic growth and welfare should be entirely ascribed to the working of the labour subsidies (taxes) rather than to the intergenerational tax transfer channel.

2.4 The Optimal Number of Children and the Savings Path. Substituting out eq. (8) into (3) and (4) to eliminate τ_t and solving for n and s respectively, yields:

$$n^*(\sigma) = \frac{\rho}{q(1+\sigma+\rho)}, \tag{9}$$

$$s_t(\sigma) = \phi \frac{1+\sigma}{1+\sigma+\rho} w_t, \tag{10}$$

where $\phi(1+\sigma)/(1+\sigma+\rho)$ represents the (constant) propensity to save. Thus, by looking at (9) and (10), it is evident that taxation policies distorts both agents' fertility choices and savings behaviour. Further, eq. (9) implies that the optimal number of children does not depend on the wage rate but only the subsidy/tax rate, and population is increasing over time ($n > 1$) if and only if $\sigma < [\rho - q(1+\rho)]/q$, that is the labour subsidy (tax) must be sufficiently small (high) for ensuring a positive long-run population growth rate.

2.5 The Long-Run Equilibrium. We now close the model to characterise the long-run equilibrium. Given the government balanced budget equation, (see eq. (8)) and the economy's resource constraint, $y_t = c_t^{\,y} + c_t^{\,o}/n_{t-1} + n_t k_{t+1}$, the market-clearing condition in goods as well as in capital markets is given by the equality between savings and investments, i.e. with $\delta = 1$ and $N_{t+1} = n_t N_t$ equilibrium implies $n_t k_{t+1} = s_t$. Substituting out for n and s according to eqs. (9) and (10), and using eq. (7), capital evolves over time according to:

$$k_{t+1} = \mu(1-\alpha)A(1+\sigma)(k_t / h_t)^{\alpha}, \tag{11}$$

where $\mu := q\phi/\rho$. Steady-state implies $k_{t+1} = k_t = k^*$. By recalling that $h = 1 - qn^*(\sigma)$, the per-capita long-run capital accumulation is the following:[7]

$$k^*(\sigma) = [\mu(1-\alpha)A(1+\sigma)]^{\frac{1}{1-\alpha}} \cdot \left[\frac{1+\sigma}{1+\sigma+\rho}\right]^{\frac{-\alpha}{1-\alpha}}. \tag{12}$$

Using eq. (12), the long-run per-capita output is given by:

$$y^*(\sigma) = (k^*(\sigma))^{\alpha} \cdot \left[\frac{1+\sigma}{1+\sigma+\rho}\right]^{1-\alpha}. \tag{13}$$

The effects of the wage subsidy/tax on both long-run capital accumulation and output are summarised in the following two propositions:[8]

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[7] As regards stability, the analysis of eq. (11) is qualitatively similar to the original Diamond's model with endogenous fertility. In particular, $\partial k_{t+1}/\partial k_t \big|_{k_t = k^*} = \alpha < 1$ implying that in the neighbourhood of the steady-state the trajectory will always be monotonic and convergent towards the equilibrium, whatever the value of the subsidy/tax rate.

[8] It is worth noting that if $\sigma = 0$ the standard results of the competitive economy without tax policies hold. The building of the Diamond OLG model with endogenous fertility as well as the derivation of its steady state outcomes in terms of capital stock ($k^*{}_c$), output ($y^*{}_c$), fertility rate ($n^*{}_c$) and welfare ($V^*{}_c$) are rather conventional and, thus, not reported here for economy of space.

Proposition 1. *Let* $\sigma > 0$ *(*$-1 < \sigma < 0$*). The T/S (S/T) policy always boosts (depresses) the long-run capital accumulation, and* $k^*(\sigma) > k^*_c$ *(*$k^*(\sigma) < k^*_c$*) for any* $\sigma > 0$ *(*$-1 < \sigma < 0$*).*
Proof. *Differentiating eq. (12) with respect to* σ *yields:*

$$\frac{\partial k^*(\sigma)}{\partial \sigma} = \begin{cases} \dfrac{k^*(\sigma)(1+\sigma+\rho(1-\alpha))}{(1-\alpha)(1+\sigma)(1+\sigma+\rho)} > 0, & if \quad \sigma > 0 \\ \dfrac{-k^*(\sigma)(1+\sigma+\rho(1-\alpha))}{(1-\alpha)(1+\sigma)(1+\sigma+\rho)} < 0, & if \quad -1 < \sigma < 0 \end{cases}.$$

Since $k^*(\sigma)$ *is a positive (negative) monotonic function of* σ *for any* $\sigma > 0$ *(*$-1 < \sigma < 0$*) and* $k^*(\sigma) = k^*_c$ *if and only if* $\sigma = 0$*, it follows that* $k^*(\sigma) > k^*_c$ *(*$k^*(\sigma) < k^*_c$*) for any* $\sigma > 0$ *(*$-1 < \sigma < 0$*). **Q.E.D.***

Thus, any increase of the wage subsidy/tax rate rises/lowers the steady-state stock of capital. Further, from Proposition 1 it emerges that under T/S (S/T) policy, the steady-state wage and interest rate are respectively higher (lower) and lower (higher) than the corresponding values in the case of non-existence of tax policies. In the long-run, the savings rate, therefore, is always increased (decreased) by the working of the T/S (S/T) policy through two positive (negative) effects: 1) a direct effect played by the subsidy (tax) rate on the propensity to save (see eq. (10)) and 2) a steady-state indirect wage effect.

Proposition 2. *Let* $\sigma > 0$ *(*$-1 < \sigma < 0$*). The T/S (S/T) policy always enhances (deteriorates) the long-run economic growth, and* $y^*(\sigma) > y^*_c$ *(*$y^*(\sigma) < y^*_c$*) for any* $\sigma > 0$ *(*$-1 < \sigma < 0$*).*
Proof. *Differentiating eq. (13) with respect to* σ *yields:*

$$\frac{\partial y^*(\sigma)}{\partial \sigma} = \begin{cases} \dfrac{y^*(\sigma)(\alpha(1+\sigma)+\rho(1-\alpha))}{(1-\alpha)(1+\sigma)(1+\sigma+\rho)} > 0, & if \quad \sigma > 0 \\ \dfrac{-y^*(\sigma)(\alpha(1+\sigma)+\rho(1-\alpha))}{(1-\alpha)(1+\sigma)(1+\sigma+\rho)} < 0, & if \quad -1 < \sigma < 0 \end{cases}.$$

Since $y^*(\sigma)$ *is a positive (negative) monotonic function of* σ *for any* $\sigma > 0$ *(*$-1 < \sigma < 0$*) and* $y^*(\sigma) = y^*_c$ *if and only if* $\sigma = 0$*, it follows that* $y^*(\sigma) > y^*_c$ *(*$y^*(\sigma) < y^*_c$*) for any* $\sigma > 0$ *(*$-1 < \sigma < 0$*). **Q.E.D.***

Proposition 2 says that any increase of the wage subsidy/tax rate always enhances/deteriorates the long-run economic growth.
The effects of the wage subsidy/tax on the long-run rate of fertility are described by the following proposition:

Proposition 3. *Let* $\sigma > 0$ *(*$-1 < \sigma < 0$*). The T/S (S/T) policy always reduces (increases) the long-run population growth rate, and* $n^*(\sigma) < n^*_c$ *(*$n^*(\sigma) > n^*_c$*) for any* $\sigma > 0$ *(*$-1 < \sigma < 0$*).*
Proof. *Differentiating eq. (9) with respect to* σ *yields:*

$$\frac{\partial n^*(\sigma)}{\partial \sigma} = \begin{cases} \dfrac{-n^*(\sigma)}{(1+\sigma+\rho)} < 0, & if \quad \sigma > 0 \\ \dfrac{n^*(\sigma)}{(1+\sigma+\rho)} > 0, & if \quad -1 < \sigma < 0 \end{cases}.$$

Since $n^*(\sigma)$ is a negative (positive) monotonic function of σ for any $\sigma > 0$ ($-1 < \sigma < 0$) and $n^*(\sigma) = n^*{}_c$ if and only if $\sigma = 0$, it follows that $n^*(\sigma) < n^*{}_c$ ($n^*(\sigma) > n^*{}_c$) for any $\sigma > 0$ ($-1 < \sigma < 0$). ***Q.E.D.***

The result stated in Proposition 3 stems directly from the role played by σ. It says that the wage subsidy/tax rate may be used as an instrument for reducing/increasing population growth even if n does not depend on the wage. It is worth to note that, although the income of the younger generation remains unchanged under both policies, these latter always have effects on demographic as well as macroeconomic variables.

2.6 The Lifetime Welfare. After having discussed the economic growth and fertility outcomes of the model we turn to the welfare analysis, which has been carried out in terms of comparing steady state paths of the lifetime welfare of the representative generation, following - among many others - Samuelson (1975). The steady-state young-adult and old-age consumption are the following:

$$c^{*y}(\sigma) = (1-\phi)(1-\alpha)A\left(\frac{1+\sigma}{1+\sigma+\rho}\right)^{1-\alpha}\left(k^*(\sigma)\right)^{\alpha}, \tag{14}$$

and

$$c^{*o}(\sigma) = \phi\alpha(1-\alpha)A^2\left(\frac{1+\sigma}{1+\sigma+\rho}\right)^{2(1-\alpha)}\left(k^*(\sigma)\right)^{2\alpha-1}. \tag{15}$$

The government's objective is to maximise the representative agent's indirect utility with respect to the wage subsidy/tax rate given his optimal choices, i.e.:

$$\max_{\{\sigma\}} V^*(\sigma) = (1-\phi)\ln\left(c^{*y}(\sigma)\right) + \phi\ln\left(c^{*o}(\sigma)\right) + \rho\ln\left(n^*(\sigma)\right). \tag{16}$$

We may then proceed to analyse the relationship among labour subsidy/tax and welfare. Differentiating (16) with respect to σ yields:

$$\frac{\partial V^*(\sigma)}{\partial\sigma} = 0 \Leftrightarrow \frac{1-\phi}{c^{*y}(\sigma)}\frac{\partial c^{*y}(\sigma)}{\partial\sigma} + \frac{\phi}{c^{*o}(\sigma)}\frac{\partial c^{*o}(\sigma)}{\partial\sigma} + \frac{\rho}{n^*(\sigma)}\frac{\partial n^*(\sigma)}{\partial\sigma} = 0, \tag{17}$$

or

$$\frac{\partial V^*(\sigma)}{\partial\sigma} = 0 \Leftrightarrow \begin{cases} \dfrac{(\alpha-\alpha_1)-\sigma(\alpha_2-\alpha)}{(1-\alpha)(1+\sigma)(1+\sigma+\rho)} = 0, & if \quad \sigma > 0 \\ -\dfrac{(\alpha-\alpha_1)-\sigma(\alpha_2-\alpha)}{(1-\alpha)(1+\sigma)(1+\sigma+\rho)} = 0, & if \quad -1 < \sigma < 0 \end{cases}. \tag{18}$$

where $\alpha_1 := \phi/(1+\phi)$ and $\alpha_2 := (\phi+\rho)/(1+\phi+\rho)$ with $\alpha_2 > \alpha_1$. The sign of (18) crucially depends on the size α (the capital's weight in technology) relative to those of ϕ and ρ (the consumption and the preference for children parameters respectively). If $\alpha > \alpha_2$, then $V^*(\sigma)$ is a monotonic function of σ, whereas provided that $\alpha < \alpha_2$, $V^*(\sigma)$ is an inverted U-shaped function. In the latter case, the value of the wage subsidy/tax rate which solves (18) is simply:

$$\sigma = \sigma_V := \frac{\alpha-\alpha_1}{\alpha_2-\alpha}. \tag{19}$$

By eqs. (9) and (14)-(19), Proposition 4 follows:

Proposition 4*. 1) Let $\alpha > \alpha_2$. The T/S (S/T) economy is welfare-preferred (welfare-worsened) with respect to the competitive economy without tax policies for any $\sigma > 0$ ($-1 < \sigma < 0$).*

2) Let $\alpha_1 < \alpha < \alpha_2$. The T/S economy is welfare-preferred to the competitive economy without tax policies for any $0 < \sigma < \sigma^{\circ}$, and the government maximises the individual's lifetime welfare if and only if $\sigma = \sigma_V$ with $\sigma_V > 0$. The S/T economy is welfare-worsened for any $-1 < \sigma < 0$.

3) Let $\alpha < \alpha_1$. The S/T economy is welfare-preferred to the competitive economy without tax policies for any $\sigma^{\circ\circ} < \sigma < 0$, and the government maximises the individual's lifetime welfare if and only if $\sigma = \sigma_V$ with $-1 < \sigma_V < 0$. The T/S economy is welfare-worsened for any $\sigma > 0$.

***Proof**. If $\alpha > \alpha_2$, $\partial V^*(\sigma)/\partial\sigma > 0$ ($\partial V^*(\sigma)/\partial\sigma < 0$) for any $\sigma > 0$ ($-1 < \sigma < 0$). Thus, $V^*(\sigma)$ is a positive (negative) monotonic function of σ for any $\sigma > 0$ ($-1 < \sigma < 0$). Knowing that $V^*(\sigma) = V^*{}_c$ if and only if $\sigma = 0$, then $V^*(\sigma) > V^*{}_c$ ($V^*(\sigma) < V^*{}_c$) for any $\sigma > 0$ ($-1 < \sigma < 0$). Thus the T/S (S/T) policy always improves (decreases) the lifetime welfare.*

Provided that $\alpha < \alpha_2$, $\partial V^*(\sigma)/\partial\sigma \underset{<}{\overset{\geq}{-}} 0$ if $\sigma \underset{>}{\overset{\leq}{-}} \sigma_V$ for any $\sigma > 0$ ($\sigma \underset{<}{\overset{\geq}{-}} \sigma_V$ for any $-1 < \sigma < 0$). This implies that $\sigma = \sigma_V$ is a global maximum of $V^*(\sigma)$ for any $-1 < \sigma < +\infty$.

Let $\alpha_1 < \alpha < \alpha_2$: by using (19) it follows that $\sigma_V > 0$. In this case, a) if $\sigma > 0$, then $V^*(\sigma)$ is a positive (negative) monotonic function of σ for any $0 < \sigma < \sigma_V$ ($\sigma > \sigma_V$), and b) if $-1 < \sigma < 0$, then $V^*(\sigma)$ is a negative monotonic function of σ. Thus, knowing that $V^*(\sigma) = V^*{}_c$ if and only if $\sigma = 0$, there exists one and only one value of the wage subsidy rate, say $\sigma = \sigma^{\circ} > \sigma_V$, such that $V^*(\sigma^{\circ}) = V^*{}_c$. Hence, if $\sigma > 0$, then $V^*(\sigma) > V^*{}_c$ for any $0 < \sigma < \sigma^{\circ}$, whilst $V^*(\sigma) < V^*{}_c$ for any $\sigma > \sigma^{\circ}$, with $V^*(\sigma) - V^*{}_c > 0$ being maximised if and only if $\sigma = \sigma_V > 0$, while if $-1 < \sigma < 0$, then $V^*(\sigma) < V^*{}_c$.

*Let $\alpha < \alpha_1$: by using (19) it follows that $-1 < \sigma_V < 0$. In this case, a) if $-1 < \sigma < 0$, then $V^*(\sigma)$ is a positive (negative) monotonic function of σ for any $-1 < \sigma < \sigma_V$ ($\sigma_V < \sigma < 0$), and b) if $\sigma > 0$, then $V^*(\sigma)$ is a negative monotonic function of σ. Thus, knowing that $V^*(\sigma) = V^*{}_c$ if and only if $\sigma = 0$, there exists one and only one value of the wage tax rate, say $\sigma = \sigma^{\circ\circ} < \sigma_V$, such that $V^*(\sigma^{\circ}) = V^*{}_c$. Hence, if $-1 < \sigma < 0$, then $V^*(\sigma) > V^*{}_c$ for any $\sigma^{\circ\circ} < \sigma < 0$ and $V^*(\sigma) < V^*{}_c$ for any $\sigma < \sigma^{\circ\circ}$, with $V^*(\sigma) - V^*{}_c > 0$ being maximised if and only if $\sigma = \sigma_V$, where $-1 < \sigma_V < 0$, while if $\sigma > 0$, then $V^*(\sigma) > V^*{}_c$. **Q.E.D.***

Therefore, provided a sufficiently high capital weight in technology ($\alpha > \alpha_2$), the lifetime welfare of the representative generation is always higher (lower) under T/S (S/T) regime than in the absence of tax policies. Moreover, the higher ϕ and ρ are, the higher the capital weight in production needed to achieve a higher welfare will be.[9] This means that the introduction of the T/S policy is favoured by a low preference for children as well as a low propensity to save. Otherwise, if the size of the capital's share in income is sufficiently small (relative to those of consumption and preference for children parameters), two cases are possible: i) moderate values of α, that is $\alpha_1 < \alpha < \alpha_2$, and ii) low values of α, that is $\alpha < \alpha_1$. In the former case, the government maximises the lifetime welfare under T/S policy by choosing an appropriate wage subsidy rate, whereas $V^*(\sigma)$ is always worsened by the introduction of the S/T policy; in the latter case, instead, welfare is maximised under S/T policy by fixing an adequate labour tax, with $V^*(\sigma)$ being always reduced by the introduction of the T/S policy.

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[9] This can be ascertained by showing that $\partial\alpha_V / \partial\phi = \partial\alpha_V / \partial\rho = 1/(1+\phi+\rho)^2 > 0$.

2.7 A Graphical Illustration. A qualitatively analysis, for parametric configurations chosen only for illustrative purposes, may help us in evaluating how income, welfare and fertility change along with the level of the wage subsidy/tax rate.

Figure 1 clearly shows that under T/S (S/T) policy the long-run income is always higher (lower) than in the case of non-existence of tax policies. Figure 2, instead, says that in the case of T/S (S/T) policy the long-run population growth rate is always lower (higher) than in the absence of taxes/subsidies. In particular, the fertility rate is always decreasing (increasing) with the wage subsidy (tax) rate. Figure 2 also shows that when the wage benefit rate is fixed at a too high level, population becomes stationary or it may even decrease.

Figure 3 displays, in the case $\alpha > \alpha_2$, that the lifetime welfare is a positive monotonic function of σ, suggesting that - if the capital's share in income is sufficiently high - the policy maker should choose a T/S policy fixing a wage subsidy as higher as possible such as to maximise the lifetime welfare as well as economic growth (see Figure 1). Overall, Figures 1-3 clearly show that the policy maker may choose the level of the labour subsidy aiming to achieve as higher as possible output and welfare levels compatible with the desired population growth rate - by keeping its budget constraint permanently balanced. Thus, for higher values of α relative to those of ϕ and ρ, the wage subsidy rate may also be used as a parameter for controlling population growth, once the desired welfare level has been decided, following the trade-off between higher welfare and low fertility implicitly involved in the different behaviours of welfare and fertility shown in Figures 2 and 3.

Besides, provided a sufficiently small capital weight in production ($\alpha < \alpha_2$) two cases are possible: i) moderate values of α, i.e. $\alpha_1 < \alpha < \alpha_2$ and ii) low values of α, i.e. $\alpha < \alpha_1$. In the former case, a welfare-maximising wage subsidy rate ($\sigma > 0$) is picked up, whereas in the latter case, the government maximises the lifetime welfare of the representative generation by choosing an appropriate wage tax ($-1 < \sigma < 0$). Figure 4 depicts case i): it says that policy maker should fix a subsidy rate of about 62% in order to maximise welfare. Finally, Figure 5 displays case ii), where, in contrast with the former case, a tax rate of about 14% is necessary to achieve the maximum welfare. This analysis suggests two interesting remarks: 1) the higher α (the lower the preference for children parameter, ρ) is, the more the introduction of a wage subsidy rate rather than a wage tax rate is beneficial for the lifetime welfare; and 2) while in the case $\alpha > \alpha_2$ the policy maker used the subsidy rate for controlling the desired levels of both welfare and population growth, in the other cases ($\alpha_1 < \alpha < \alpha_2$ and $\alpha < \alpha_1$), where a unique welfare-maximising policy does exist, the rate of population growth will univocally be determined by the welfare-maximising subsidy (tax) rate. Hence, for moderate values of α ($\alpha_1 < \alpha < \alpha_2$), the government maximises $V^*(\sigma)$ under T/S policy with a corresponding relatively low population growth rate, whereas provided a sufficiently small value of α ($\alpha < \alpha_1$), welfare is maximised under S/T policy with a corresponding relatively high population growth rate.

[Figures 1 - 5 about here]

2.8 Optimality. As in Samuelson (1975), and following also Abio (2003), optimality is defined here as the allocation that maximises the steady state utility of the representative individual subject to the economy's resource constraint. In particular, the social planner faces with the problem of maximising the following logarithmic felicity function:

$$\max_{\{c^y, c^o, n, k\}} U(c^y, c^o, n) = (1-\phi)\ln(c^y) + \phi\ln(c^o) + \rho\ln(n),$$

subject to

$$Ak^{\alpha}h^{1-\alpha} = c^y + c^o/n + nk,$$

where $h = 1 - qn$.

The first order conditions for an optimal interior solution are:

$$\frac{c^o}{c^y}=\frac{\phi}{1-\phi}n\,, \tag{20}$$

$$\frac{c^o}{n^2}+\frac{c^y}{n}\frac{\rho}{1-\phi}=k+q(1-\alpha)A\left(\frac{k}{1-qn}\right)^{\alpha}, \tag{21}$$

$$\alpha A\left(\frac{k}{1-qn}\right)^{\alpha-1}=n\,. \tag{22}$$

Eq. (20) gives the optimal allocation between consumption of young-adult and old-age individuals. It equates the social marginal utility of current and future consumption in terms of current consumption. Eq. (21) determines the optimal number of children by equating benefits and costs of a marginal increase in the population growth rate. In particular, the left-hand side of (21) gives the marginal benefit of an increase in the number of children, while the right-hand side says that the marginal costs of population growth must be split among two terms: the first term represents the capital that must be expanded in order for the capital-labour ratio to be maintained, plus the production loss due to having supposed a time cost to rearing children. Finally, eq. (22) determines the golden-rule stock of capital.

Exploiting the first order conditions (20)-(22) and using the economy's resource constraint yields, respectively, the golden-rule number of children as a function of the basic parameters of the model, the golden-rule per-capita stock of capital and the optimal social values of both young-adult and old-age consumption, that is:

$$n_{GR}=\frac{\alpha_2-\alpha}{q(\alpha_3-\alpha)}, \tag{23}$$

$$k_{GR}=\left(\frac{\alpha A}{n}\right)^{\frac{1}{1-\alpha}}\cdot(1-qn_{GR}), \tag{24}$$

$$c^y{}_{GR}=(1-\phi)\left(\frac{1-\alpha}{\alpha}\right)n_{GR}\cdot k_{GR}\,, \tag{25}$$

$$c^o{}_{GR}=\phi\left(\frac{1-\alpha}{\alpha}\right)n^2{}_{GR}\cdot k_{GR}, \tag{26}$$

where $\alpha_3 := (1+\phi+\rho)/(2+\phi+\rho)$ with $\alpha_3>\alpha_2$.[10]

The constrained maximisation of the representative individual's indirect utility (see eq. (16)) may be interpreted as a Stackelberg game played by the benevolent government in the decentralised economy. In particular, the government is a Stackelberg leader with respect to individuals and firms and, given the values of both periods consumption as well as the number of children chosen optimally by individuals, together with the marginal productivity conditions on capital an labour obtained by the representative firm, and knowing also that the lump-sum taxes (subsidies) on the income of the young-adult agents is an endogenous variable, it chooses a value of the wage subsidy (tax) rate such as to maximise welfare, that is:

$$\max_{\{\sigma\}} V^*(\sigma)=(1-\phi)\ln(c^{*y}(\sigma))+\phi\ln(c^{*o}(\sigma))+\rho\ln(n^*(\sigma)), \tag{27}$$

subject to

$$\tau=\sigma wh\,,$$

$$c^{*y}(\sigma)=\frac{1-\phi}{1+\rho}[w(1+\sigma)-\tau],$$

$$c^{*o}(\sigma)=\frac{\phi}{1+\rho}(1+r)[w(1+\sigma)-\tau],$$

[10] Note that if ϕ and ρ approach zero, $\alpha_3=1/2$. Thus, $\alpha_3>1/2$ for any $\phi,\rho>0$.

$$n^*(\sigma)=\frac{\rho}{1+\rho}\left[\frac{w(1+\sigma)-\tau}{qw(1+\sigma)}\right],$$

where we recall $h=1-qn^*(\sigma)$, with r and w being determined by eqs. (6) and (7) respectively. Differentiating (27) with respect to σ yields:

$$\frac{\partial V^*(\sigma)}{\partial \sigma}=0 \Leftrightarrow \sigma=\sigma_V:=\frac{\alpha-\alpha_1}{\alpha_2-\alpha}. \tag{28}$$

The value of the wage subsidy/tax rate as given by (28) is exactly that one which solves the government Stackelberg game.
We can now give the following proposition:

Proposition 5*. 1) Provided that $\alpha_1<\alpha<\alpha_2$ ($\alpha<\alpha_1$), the optimal population growth rate (OPGR henceforth), n_{GR}, is achieved by implementing a T/S (S/T) policy. It is determined by the following wage subsidy (tax) rate:*

$$\sigma_{nGR}=\frac{\alpha-\alpha_1}{\alpha_2-\alpha}. \tag{29}$$

2) Given the solution of the Stackelberg game (see eq. (28)), the optimal wage subsidy/tax rate is exactly that one which maximises the lifetime welfare in the decentralised economy, that is:

$$\sigma_V=\sigma_{nGR}. \tag{30}$$

Proof*. 1) The proof straightforwardly derives by subtracting eq. (23) from eq. (9) and solving for σ. As regards point 2) the proof follows directly by looking at (28) and (29).* ***Q.E.D.***

Eq. (30) gives the optimal value of the wage subsidy/tax rate such that $n_{GR}=n^*(\sigma)$. Thus, provided that $\alpha_1<\alpha<\alpha_2$ ($\alpha<\alpha_1$) a benevolent government is always able to achieve the golden-rule number of children by implementing a welfare-maximising T/S (S/T) policy, fixing the subsidy (tax) rate at $\sigma=\sigma_V$.
Given eqs. (24)-(26) the indirect utility function

$$V(n)=(1-\phi)\ln\left(c^y{}_{GR}\right)+\phi\ln\left(c^o{}_{GR}\right)+\rho\ln(n),$$

is maximised if and only if

$$\frac{\partial V(n)}{\partial n}=0 \Leftrightarrow n=n_{GR}.$$

If $\alpha>\alpha_2$, then $V(n)$ is a negative monotonic function of the number of children for any $n>0$ (see Figures 6 and 3). In this case, as it can be seen by looking at Figure 6, the optimal number of children is $n_{GR}=0$. On the contrary, provided that $\alpha<\alpha_2$, $V(n)$ is an inverted U-shaped function of n with $V(n)$ being maximised if and only if $n=n_{GR}$. In this latter case, it emerges that for moderate (low) values of the capital's share in income, i.e. $\alpha_1<\alpha<\alpha_2$ ($\alpha<\alpha_1$), under T/S (S/T) policy, a benevolent government is able to pick up $n^*(\sigma)=n_{GR}$ by setting the welfare-maximising wage subsidy (tax) at the rate $\sigma=\sigma_V>0$ ($\sigma=\sigma_V$ with $-1<\sigma_V<0$). For a graphical illustration of the comparison of both positive and normative analyses, see Figures 4 and 7 respectively as regards the case $\alpha_1<\alpha<\alpha_2$ (T/S policy), and Figures 5 and 8 respectively for the case $\alpha<\alpha_1$ (S/T policy).

[Figures 6 - 8 about here]

The remarkable result is that *the benevolent government (Stackelberg game), which does not have the OPGR target, by choosing the welfare-maximising wage subsidy (tax) rate in the decentralised economy, behaves as if it were the social planner as regards the achievement of the OPGR.*

In other terms, the T/S - S/T policies implemented by the benevolent government not such as to achieve n_{GR}, but for obtaining the maximum welfare allowing individuals to freely choose their optimal number of children, are always capable to pick up a OPGR value as well, provided that the capital's weight in production is not too high.
Note that our findings imply that to achieve the goal of the optimal social number of children is not necessary to assume the presence of a social planner having an explicit OPGR target (e.g., a dictatorial imposition of the optimal social number of children, see for instance the case of China), but, on the contrary, it is sufficient the intervention of a benevolent government with the decentralised individual's lifetime utility target through simple intra-generational redistributionary tax policies on the income of the younger generation. When the wage subsidy/tax rate is fixed at its welfare-maximising value, the number of children freely chosen by individuals is exactly that one chosen by the social planner. This does not raise any ethic concern involved with the birth control issue.

3 Concluding Comments

This paper has examined the steady state consequences (on the long-run income and the lifetime welfare) of the introduction of intra-generational tax-cum-subsidy/subsidy-cum-tax policies in a basic OLG neoclassical growth context, in particular focusing on the role of endogenous fertility choices. We have found, surprisingly, that even if the policies we have investigated do not involve any saving-enhancing inter-generational tax transfer (as it would have been the case with capital income taxes which redistribute resources from the old people to young individuals (savers)), the working of such policies may boost (deteriorates) economic growth. In particular, the long-run economic growth is always higher (lower) when a tax-cum-subsidy (subsidy-cum-tax) policy is implemented.
As regards the lifetime welfare, either a subsidy rate or a tax rate may be welfare-preferred depending on the relative values of technology as well as preference parameters (in general, the higher the capital weight in production - the lower the preference for children - is, the more a subsidy rate instead of a tax rate is welfare preferred). Further, provided that the capital's weight in production is sufficiently high (low), a welfare-maximising labour subsidy (tax) rate is picked up. In particular, if the size of α relative to those of the consumption rate of time preference and the parents' preference for children parameters is high, individual's welfare is always bettered-off by introducing proportional-to-wage labour subsidies (financed with lump-sum taxes on the income of the young-adult individuals) rather than proportional-to-wage labour taxes (used for the financing of lump-sum subsidies on the income of the young-adult people), the latter always lowering it than in the case of non-existence of tax policies. On the contrary, either moderate or low values of α relative to those of ϕ and ρ suggest two precise policy prescriptions: in the former case, the lifetime welfare is maximised by choosing a wage subsidy rate, whilst the latter one prescribes that the government must fix an appropriate labour tax rate such as to achieve a welfare maximum.
Finally, as regards fertility rates, the tax policies, although leaving unchanged the parents' income, are surprisingly capable to determined the population growth rate which will be respectively higher (lower) under wage tax (subsidy) policy as in the absence of taxes. In addition, provided sufficiently high values of α, the wage-subsidy rate may be used for controlling population growth trading off between higher welfare and lower fertility.
Finally, it should be remarked that in this paper we have also performed a normative analysis of the effects of proportional labour subsidy/tax policies. In particular, provided moderate (low) values of the capital's weight in technology, a tax-cum-subsidy (subsidy-cum-tax) policy aiming to maximise the utility index of the representative generation (decentralised economy), is able to attain the optimal population growth rate as well (social planner economy). Although, for instance for ethical reasons, a government does not pursue the goal of OPGR (as, on the contrary, it is the case with the Chinese's social planner), but it only aims to maximise the utility index resulting by the optimal individual choices rather than the optimal social choices, it obtains, surprisingly, the OPGR value.

We have sought to clarify these theoretical findings using as parsimonious a model as possible. Finally, note that our results may have important applications to government policies for economic growth and welfare.

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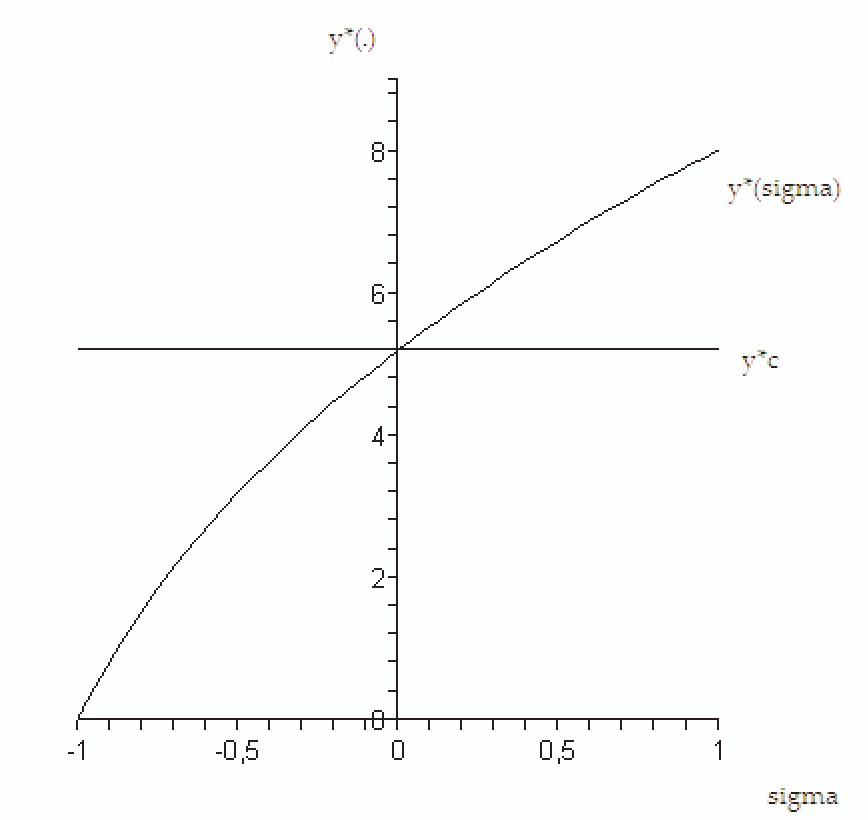

Figure 1. *The long-run income in both competitive-wage (* y^*_c *) and wage-subsidised/taxed (* $y^*(\sigma)$*) economies. Parameter values:* $A = 10$, $\alpha = 0.33$, $\phi = 0.30$, $\rho = 0.50$ *and* $q = 0.10$.

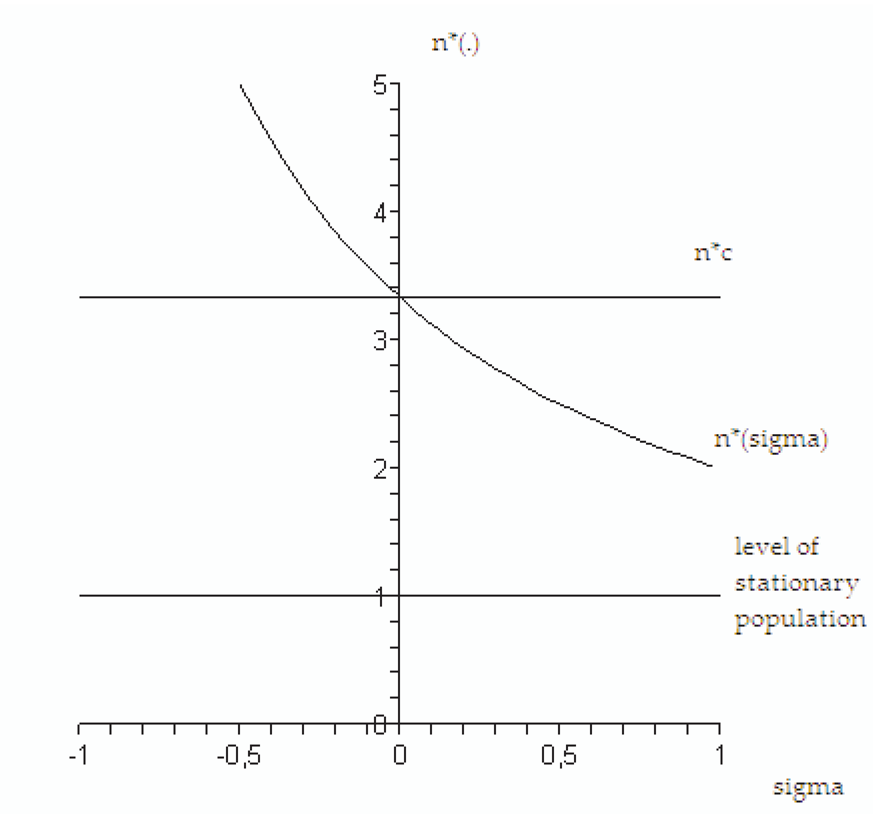

Figure 2. *The long-run population growth rate in both competitive-wage (*n^*_c*) and wage-subsidised/taxed (*$n^*(\sigma)$*) economies. Parameter values:* $A = 10$, $\alpha = 0.33$, $\phi = 0.30$, $\rho = 0.50$ *and* $q = 0.10$.

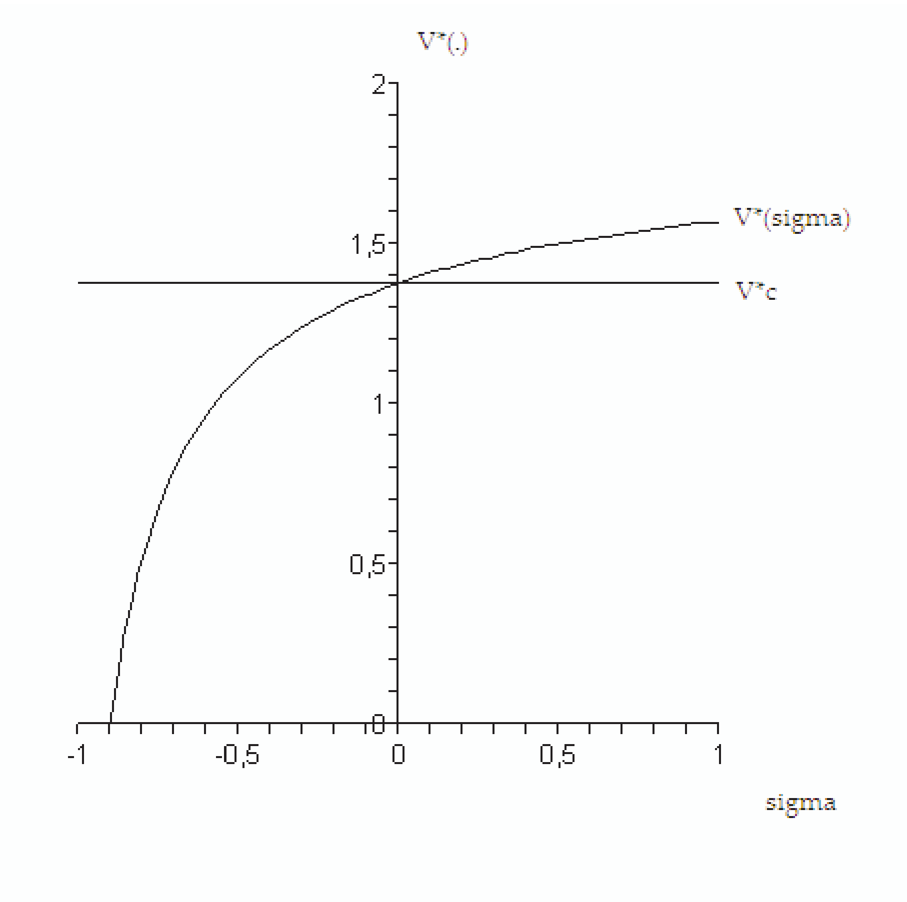

Figure 3. *Case* $\alpha > \alpha_2$. *The long-run lifetime welfare in both competitive-wage (*V^*_c*) and wage-subsidised/taxed (*$V^*(\sigma)$*) economies. Parameter values:* $A = 10$, $\alpha = 0.45$, $\phi = 0.30$, $\rho = 0.50$ *and* $q = 0.10$.

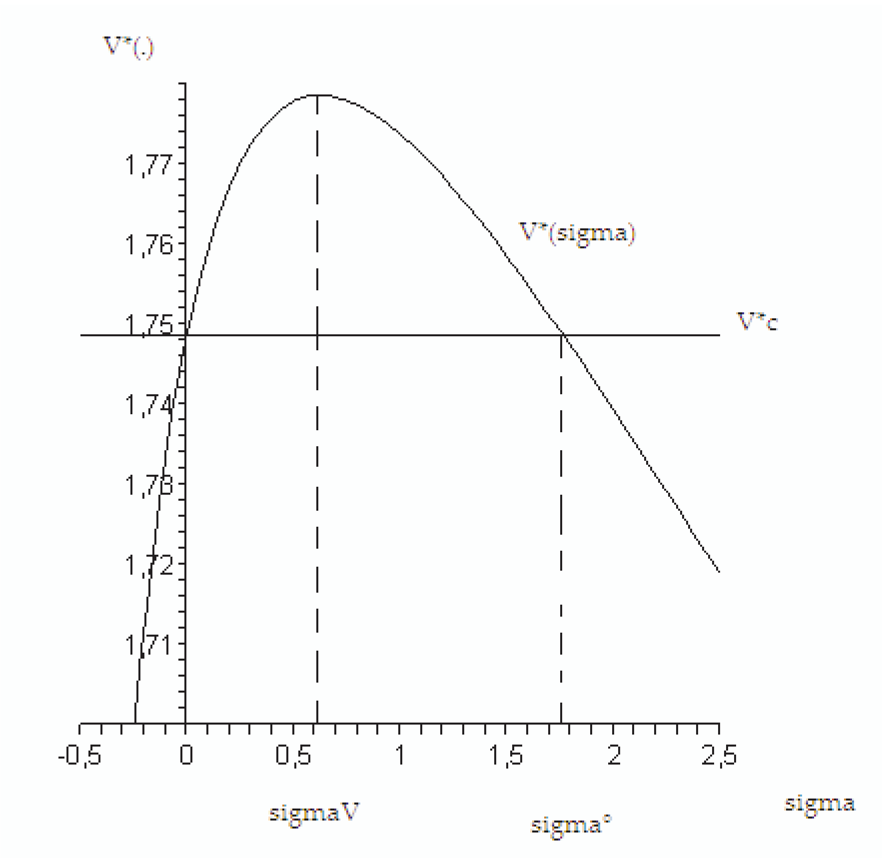

Figure 4. *Case* $\alpha_1 < \alpha < \alpha_2$. *The long-run lifetime welfare in both competitive-wage* ($V^*{}_c$) *and wage-subsidised/taxed* ($V^*(\sigma)$) *economies. Parameter values:* $A = 10$, $\alpha = 0.33$, $\phi = 0.30$, $\rho = 0.50$ *and* $q = 0.10$. *The welfare-maximised wage subsidy rate is* $\sigma_V = 0.626$.

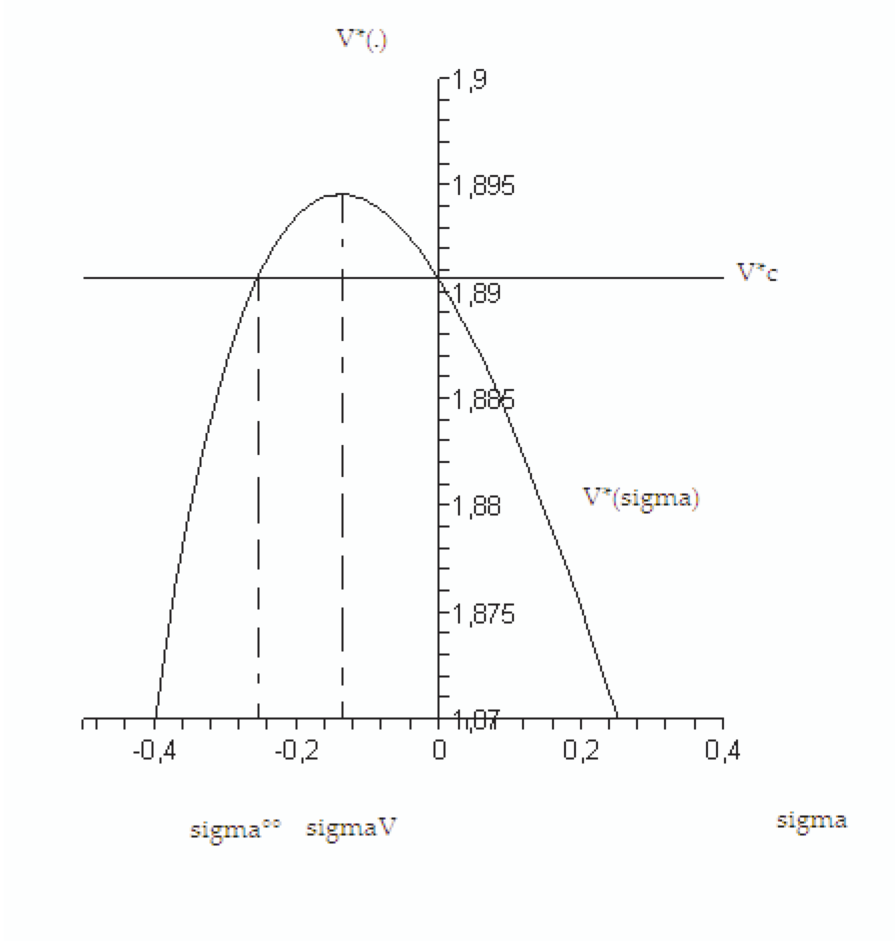

***Figure 5**. Case $\alpha < \alpha_1$. The long-run lifetime welfare in both competitive-wage (V^*_c) and wage-subsidised/taxed ($V^*(\sigma)$) economies. Parameter values: $A = 10$, $\alpha = 0.18$, $\phi = 0.30$, $\rho = 0.50$ and $q = 0.10$. The welfare-maximised wage tax rate is $\sigma_V = -0.138$.*

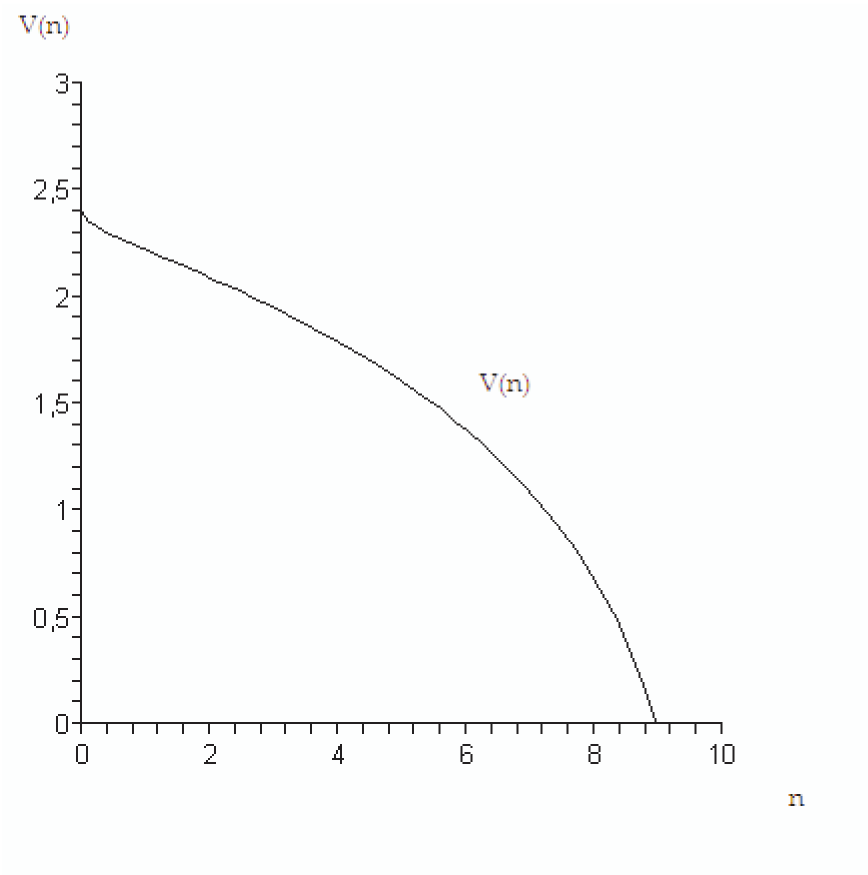

Figure 6. *Case* $\alpha > \alpha_2$. *The lifetime welfare as a function of the number of children. Parameter values:* $A = 10$, $\alpha = 0.45$, $\phi = 0.30$, $\rho = 0.50$ *and* $q = 0.10$. *The lifetime welfare is a negative monotonic function of the number of children.*

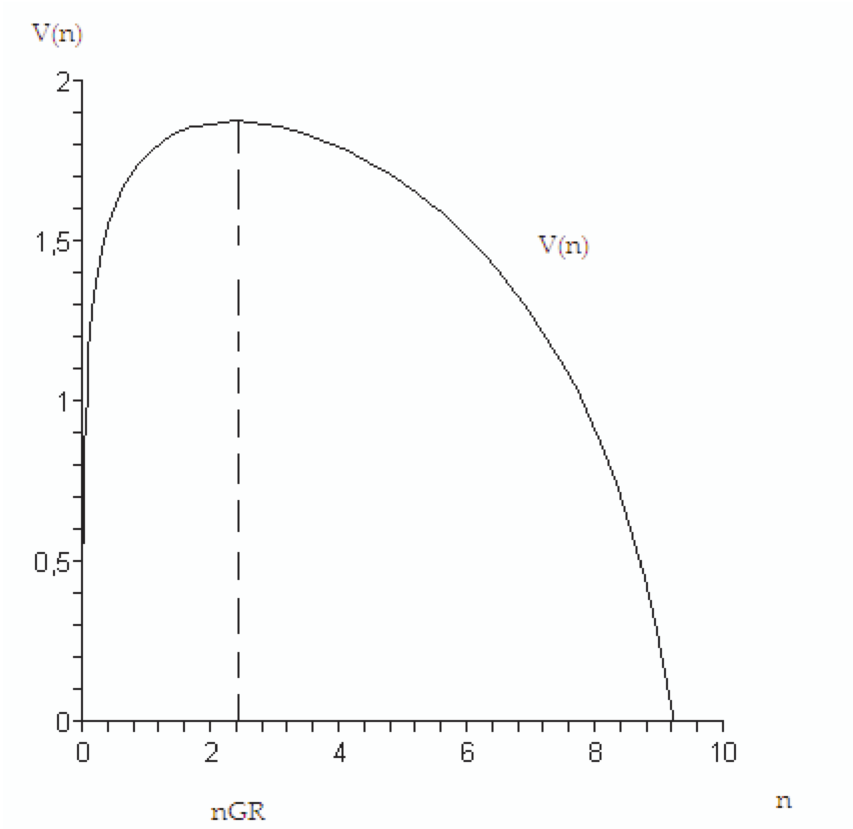

Figure 7. *Case* $\alpha_1 < \alpha < \alpha_2$. *The lifetime welfare as a function of the number of children. Parameter values:* $A = 10$, $\alpha = 0.33$, $\phi = 0.30$, $\rho = 0.50$ *and* $q = 0.10$. *The welfare-maximised population growth rate* $n_{GR} = 2.35$. *If the wage subsidy rate is fixed at its optimal value* $\sigma = \sigma_{GR} = \sigma_V = 0.626$, *then* $n^*(\sigma) = n_{GR} = 2.35$.

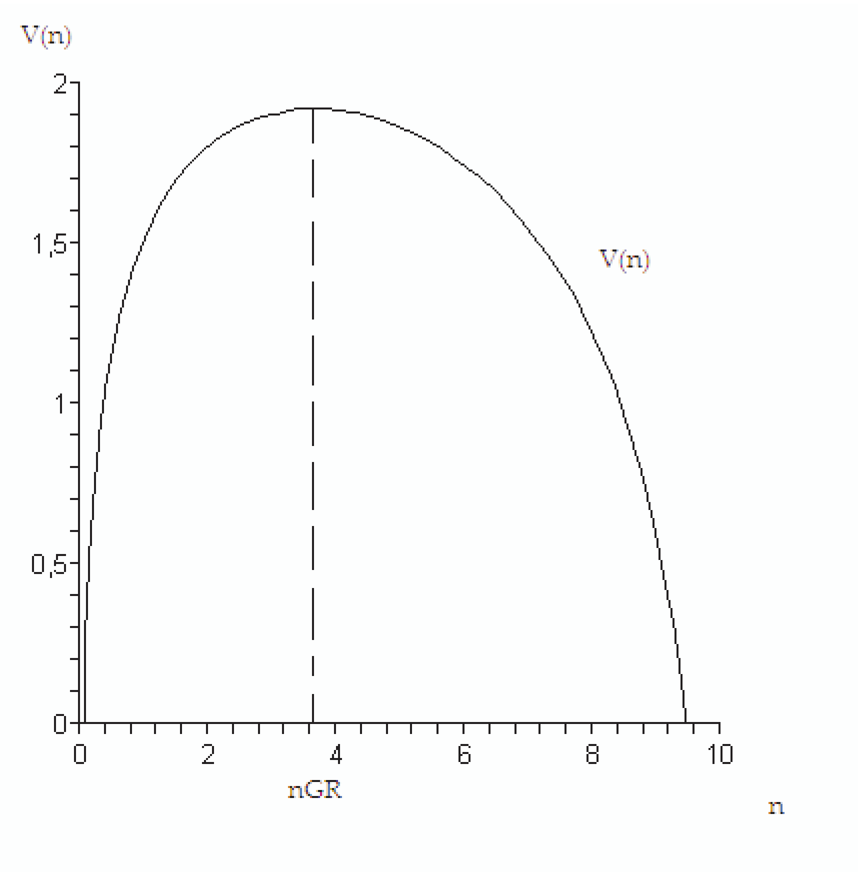

Figure 8. *Case* $\alpha < \alpha_1$. *The lifetime welfare as a function of the number of children. Parameter values:* $A = 10$, $\alpha = 0.18$, $\phi = 0.30$, $\rho = 0.50$ *and* $q = 0.10$. *The welfare-maximised population growth rate* $n_{GR} = 3.67$. *If the wage tax rate is fixed at its optimal value* $\sigma = \sigma_{GR} = \sigma_V = -0.138$, *then* $n^*(\sigma) = n_{GR} = 3.67$.